The finite-time Gerber-Shiu penalty function for two classes of risk processes

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The finite – time Gerber – Shiu penalty function for two classes of risk processes

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Outline

Consider that the surplus at time t of an insurance company is given by

$$U(t) = u + ct - S(t), \quad t \geq 0,$$

where

- $u = U(0) \geq 0$ is the *initial capital*;
- $c > 0$ is the constant *premium income* per unit time;
- $S(t)$ is the *aggregate claims amount* up to time t and $S(0) = 0$.

Let τ denote the time of ruin, so that

$$\tau = \begin{cases} \inf\{t \geq 0 : U(t) < 0 \,|\, U(0) = u\}, \\ \infty, \text{ if } U(t) \geq 0 \text{ for all } t > 0. \end{cases}$$

The corresponding probability of ultimate ruin is

$$\psi(u) = \mathbb{P}(\tau < \infty \,|\, U(0) = u),$$

and the survival (non-ruin) probability is $\phi(u) = 1 - \psi(u)$.
The finite-time ruin probability of the company up to time t is

$$\psi(u, t) = \mathbb{P}(\tau < t \,|\, U(0) = u).$$

Gerber and Shiu (1998) introduced the concept of expected discounted penalty function (EDPF), defined as

$$m(u) = \mathbb{E}\left[e^{-\delta\tau} w(U(\tau-), |\,U(\tau)\,|)\mathbb{I}(\tau < \infty) \,|\, U(0) = u\right],$$

where

- $\delta \geq 0$ is interpreted as the force of interest
- $U(\tau-)$ is the surplus immediately before ruin
- $|\,U(\tau)\,|$ is the deficit at ruin
- $w(x, y)$ is a non-negative bivariate function of x, $y \geq 0$
- $\mathbb{I}(A)$ is the indicator function of event A.

The function $w(U(\tau-), |\,U(\tau)\,|)$ can be interpreted as the "penalty" at the time of ruin.

Correlated aggregate claims risk model

- Many authors have studied continuous-time risk models involving two classes of claims.
- The approach to modeling dependent classes of business by incorporating a common component into each of the associated claim-number processes has been studied by many authors, for example, *Ambagaspitiya* (1998), *Cossette and Marceau* (2000), *Wang and Yuen* (2005).

Yuen et al. (2002) introduced a correlated risk model process involving two dependent classes of insurance risks in which the claim number processes are Poisson and Erlang(2) processes, respectively.

More specifically,

$$S(t) = \sum_{i=1}^{N_1(t)} X_i + \sum_{i=1}^{N_2(t)} Y_i,$$

where the claim number processes are correlated in the way that

$$N_1(t) = M_1(t) + M(t) \text{ and } N_2(t) = M_2(t) + M(t),$$

with $M_1(t)$, $M_2(t)$ and $M(t)$ being three independent processes.

- $M_i(t)$ is a Poisson(λ_i) process for $i = 1, 2$;
- $M(t)$ is an Erlang(2) process with parameter λ, that is, the claim inter-arrival times for $M(t)$ are independent and have Erlang(2,λ) distribution with the density function

$$k(t) = \lambda^2 t e^{-\lambda t}, \quad \text{for} \quad t > 0;$$

- $\{X_i, i \geq 1\}$ and $\{Y_i, i \geq 1\}$ are independent claim size random variables, and independent of $N_1(t)$ and $N_2(t)$.

Then, the surplus can be rewritten as

$$U'(t) = u + ct - \sum_{i=1}^{M_{12}(t)} X_i' - \sum_{i=1}^{M(t)} Y_i',$$

where

- $M_{12}(t) = M_1(t) + M_2(t)$ is still a Poisson $(\lambda_1 + \lambda_2)$ process;
- $\{X_i', i \geq 1\}$ and $\{Y_i', i \geq 1\}$ are independent random variables:

$$F_{X'}(x) = \frac{\lambda_1}{\lambda_1 + \lambda_2} F_X(x) + \frac{\lambda_2}{\lambda_1 + \lambda_2} F_Y(x),$$
$$F_{Y'}(x) = F_X(x) * F_Y(x),$$

  where $F_X * F_Y$ stands for the convolution of F_X and F_Y.
- X_i' and Y_i' are independent of $M_{12}(t)$ and $M(t)$.

Since the transformed process $U'(t)$ and the original process $U(t)$ are identically distributed, the process $U(t)$ can be examined via $U'(t)$.

- Yuen et al. (2002) derived explicit expressions for the ultimate survival (ruin) probabilities when the claim sizes are exponentially distributed and examined the asymptotic property of the ruin probability with general size distributions.
- Liu et al. (2006) derived expressions for
  -the *distribution of the surplus immediately before ruin*,
  -the *distribution of the surplus immediately after ruin*,
  -and the *joint distribution of the surplus immediately before ruin and the deficit at ruin*.
- Li and Garrido (2005) derived expressions for the survival probabilities assuming Poisson and generalized Erlang(2) processes.

The Gerber-Shiu EDPF has been extensively studied assuming a risk model with two independent classes of insurance risks. For example,

- Li and Lu (2005)-Poisson and generalized Erlang(2) processes
- Zhang et al. (2009)-Poisson and generalized Erlang(n) processes
- Wu (2009)-independent Erlang(2) processes

Assume a risk model with two independent classes of insurance risks, namely

$$U(t) = u + ct - \sum_{i=1}^{N_1(t)} X_i - \sum_{i=1}^{N_2(t)} Y_i, \quad t \geq 0,$$

where

- $\{X_i\}_{i\geq 1}$ are independent and identically distributed (i.i.d.) positive random variables with common distribution function F, density f and finite mean $E[X_i]$;
- $\{Y_i\}_{i\geq 1}$ are independent and identically distributed (i.i.d.) positive random variables with common distribution function G, density g and finite mean $E[Y_i]$;

- $\{N_1(t) : t \geq 0\}$ is a Poisson process with parameter λ and the corresponding claim inter-arrival times $\{T_i\}_{i\geq 1}$ are independent and exponentially distributed with mean $1/\lambda$.
- $\{N_2(t) : t \geq 0\}$ is a generalized Erlang (n) process and the corresponding claim inter-arrival times $\{L_i\}_{i\geq 1}$ are independent and generalized Erlang (n) that is,

$$L_i = L_{i1} + L_{i2} + ... + L_{in}, \quad i \geq 1,$$

with $\{L_{ij}\}_{i\geq 1}$ $(j = 1, 2, ..., n)$ being i.i.d. exponentially distributed random variables with mean $1/\lambda_j$.

- $\{X_i\}_{i\geq 1}$ and $\{Y_i\}_{i\geq 1}$ are independent claim size random variables and independent of $N_1(t)$, $N_2(t)$.

The Gerber-Shiu expected discounted penalty function (EDPF) in a finite time horizon is defined as

$$m(u,t) = \mathbb{E}\left[e^{-\delta\tau} w(U(\tau-), \,|\, U(\tau) \,|\,) \mathbb{I}(\tau < t) \,|\, U(0) = u\right]$$

for a fixed $t \geq 0$. If, for example,

- $\delta = 0$ and $w(x,y) = 1$ for all x and y, then $m(u,t) = \psi(u,t)$
- $\delta = 0$ and $w(x_1, y_1) = \mathbb{I}_{[0,x]}(x_1)\mathbb{I}_{[0,y]}(y_1)$, then $m(u,t)$ is

$$\mathbb{P}\left(U(\tau-) \leq x, \,|\, U(\tau) \,|\, \leq y, \tau < t \,|\, U(0) = u\right)$$

Hence, $\mathbb{P}\left(\,|\, U(\tau) \,|\, \leq y, \tau < t \,|\, U(0) = u\right)$ and $\mathbb{P}\left(\,|\, U(\tau) \,|\, \leq y \,|\, \tau < t, U(0) = u\right)$ can also be computed.

Review

- For the compound Poisson model when the claim sizes are exponentially distributed, Kocetova and Siaulys (2010) derived expressions in terms of infinite series for:

$$m(u,t) = \mathbb{E}\left[e^{-\delta\tau}\mathbb{I}(\tau < t) \,|\, U(0) = u\right],$$

$$m(u,t) = \mathbb{E}\left[\tau^k e^{-\delta\tau}\mathbb{I}(\tau < t) \,|\, U(0) = u\right], k = 1, 2, ...,$$

$$m(u,t) = \mathbb{E}\left[e^{-\delta\tau}\mathbb{I}(t_1 < \tau < t_2) \,|\, U(0) = u\right], \; t_1, t_2 > 0.$$

- Assuming that the net aggregate cash inflow is modeled by a spectrally negative Levy process, Kuznetsov and Morales (2011), obtained an expression for the Laplace transform in the t-variable of the finite-time EDPF in terms of the infinite-time EDPF and computing the finite-time EDPF is equivalent to inverting numerically the Laplace transform.
- A different definition for a finite-time EDPF is discussed in Garrido, Cojocaru and Zhou (2014).

- By considering an ordinary renewal risk model, Garcia (2010) derived explicit expressions for the finite-time survival probability, $\phi(u, t)$, through a Maclaurin series expansion.

Consider now the modified claim number processes $\{N_{2,(j-1)}(t) : t \geq 0\}$ of $\{N_2(t) : t \geq 0\}$ for $1 \leq j \leq n$, where the time until the first claim is modeled as

$$L_{1,(j-1)} = L_{1j} + L_{1j+1} + ... + L_{1n},$$

while the others are the same as that in $\{N_2(t) : t \geq 0\}$. Let us define $m_{(j-1)}(u, t)$ the finite-time Gerber-Shiu function associated to the risk process $U_{(j-1)}(t)$ obtained from $U(t)$ by replacing $N_2(t)$ with $N_{2,(j-1)}(t)$.
Note that

$$N_{2,(0)}(t) = N_2(t),\ U_{(0)}(t) = U(t),\ L_{1,(0)}(t) = L_1,\ m_{(0)}(u, t) = m(u, t).$$

Theorem

For each $j = 1, 2, ..., n$, the finite-time Gerber-Shiu function associated to the risk process $U_{(j-1)}(t)$ can be written as

$$m_{(j-1)}(u,t) = \sum_{k=1}^{\infty} \frac{t^k}{k!} Q_{k,(j-1)}(u,0), \; j = 1, 2, ..., n,$$

where $Q_{k,(j-1)}(u,0) = \frac{\partial^k m_{(j-1)}(u,t)}{\partial t^k}|_{t=0}$ *and* $m_{(0)} = m$.

Theorem (continued)

Moreover, for $j = 1, 2, ..., n-1$,

$$Q_{k,(j-1)}(u,t) = \lambda_j e^{-(\lambda+\lambda_j+\delta)t} Q_{k-1,(j)}(u+ct, 0)$$

$$+\lambda e^{-(\lambda+\lambda_j+\delta)t} \int_0^{u+ct} Q_{k-1,(j-1)}(u+ct-x, 0) f(x) dx$$

$$+\frac{\partial}{\partial t} Q_{k-1,(j-1)}(u,t),$$

Theorem (continued)

and for $j = n$,

$$Q_{k,(n-1)}(u,t) = \lambda_n e^{-(\lambda+\lambda_n+\delta)t} \int_0^{u+ct} Q_{k-1,(0)}(u+ct-x,0)g(x)dx$$

$$+\lambda e^{-(\lambda+\lambda_n+\delta)t} \int_0^{u+ct} Q_{k-1,(n-1)}(u+ct-x,0)f(x)dx$$

$$+\frac{\partial}{\partial t} Q_{k-1,(n-1)}(u,t)$$

Theorem (continued)

with starting values for $j = 1, 2, ..., n-1$,

$$Q_{1,(j-1)}(u,t) = \lambda e^{-(\lambda+\lambda_j+\delta)t} \int_{u+ct}^{\infty} w(u+ct, x-u-ct) f(x) dx$$

and

$$Q_{1,(n-1)}(u,t) = \lambda_n e^{-(\lambda+\lambda_n+\delta)t} \int_{u+ct}^{\infty} w(u+ct, x-u-ct) g(x) dx$$

$$+\lambda e^{-(\lambda+\lambda_n+\delta)t} \int_{u+ct}^{\infty} w(u+ct, x-u-ct) f(x) dx.$$

Proof. Conditioning on $M_j = \min(T_1, L_{1j})$, for $1 \leq j \leq n-1$, we may write

$$m_{(j-1)}(u,t) = \int_0^t e^{-\delta y} P(M_j = L_{1j} = y) m_{(j)}(u+cy, t-y) dy$$

$$+ \int_0^t e^{-\delta y} P(M_j = T_1 = y) \Big[\int_0^{u+cy} m_{(j-1)}(u+cy-x, t-y) f(x) dx$$

$$+ \int_{u+cy}^{\infty} w(u+cy, x-u-cy) f(x) dx \Big] dy.$$

By conditioning on $M_n = \min(T_1, L_{1n})$,

$$m_{(n-1)}(u,t) = \int_0^t e^{-\delta y} P(M_n = L_{1n} = y) \Big[\int_0^{u+cy} m(u+cy-x, t-y) g(x) dx$$

$$+ \int_{u+cy}^{\infty} w(u+cy, x-u-cy) g(x) dx \Big] dy$$

$$+ \int_0^t e^{-\delta y} P(M_n = T_1 = y) \Big[\int_0^{u+cy} m_{(n-1)}(u+cy-x, t-y) f(x) dx$$

$$+ \int_{u+cy}^{\infty} w(u+cy, x-u-cy) f(x) dx \Big] dy.$$

For $j = 1, 2, ..., n$,

$$P(M_j = T_1) = \frac{\lambda}{\lambda + \lambda_j},$$

$$P(M_j = L_{1j}) = \frac{\lambda_j}{\lambda + \lambda_j},$$

$$P(M_j > y | M_j = T_1) = P(M_j > y | M_j = L_{1j}) = e^{-(\lambda+\lambda_j)y}.$$

For $j = 1, 2, ..., n-1$,

$$m_{(j-1)}(u,t) = \lambda_j \int_0^t e^{-(\lambda+\lambda_j+\delta)y} m_{(j)}(u+cy, t-y)dy$$

$$+\lambda \int_0^t e^{-(\lambda+\lambda_j+\delta)y} \Big[\int_0^{u+cy} m_{(j-1)}(u+cy-x, t-y) f(x)dx$$

$$+ \int_{u+cy}^{\infty} w(u+cy, x-u-cy) f(x)dx \Big] dy,$$

with $m_{(0)} = m$,

and for $j = n$,

$$m_{(n-1)}(u,t) = \lambda_n \int_0^t e^{-(\lambda+\lambda_n+\delta)y} \Big[\int_0^{u+cy} m(u+cy-x, t-y) g(x) dx$$

$$+ \int_{u+cy}^{\infty} w(u+cy, x-u-cy) g(x) dx \Big] dy$$

$$+\lambda \int_0^t e^{-(\lambda+\lambda_n+\delta)y} \Big[\int_0^{u+cy} m_{(n-1)}(u+cy-x, t-y) f(x) dx$$

$$+ \int_{u+cy}^{\infty} w(u+cy, x-u-cy) f(x) dx \Big] dy.$$

Assume that $N_2(t) \sim Generalized\ Erlang(2)$. In this case, the finite-time Gerber-Shiu functions can be written as

$$m(u,t) = \sum_{k=1}^{\infty} \frac{t^k}{k!} Q_k(u,0),$$

$$m_1(u,t) = \sum_{k=1}^{\infty} \frac{t^k}{k!} Q_{k,(1)}(u,0),$$

where $Q_k(u,0)$ and $Q_{k,(1)}(u,0)$ are given by the following recursive equations

$$Q_k(u,t) = \lambda_1 e^{-(\lambda+\lambda_1+\delta)t} Q_{k-1,(1)}(u+ct,0)$$

$$+\lambda e^{-(\lambda+\lambda_1+\delta)t} \int_0^{u+ct} Q_{k-1}(u+ct-x,0) f(x)dx + \frac{\partial}{\partial t} Q_{k-1}(u,t),$$

$$Q_{k,(1)}(u,t) = \lambda_2 e^{-(\lambda+\lambda_2+\delta)t} \int_0^{u+ct} Q_{k-1}(u+ct-x,0) g(x)dx$$

$$+\lambda e^{-(\lambda+\lambda_2+\delta)t} \int_0^{u+ct} Q_{k-1,(1)}(u+ct-x,0) f(x)dx + \frac{\partial}{\partial t} Q_{k-1,(1)}(u,t),$$

with

$$Q_1(u,t) = \lambda e^{-(\lambda+\lambda_1+\delta)t} \int\limits_{u+ct}^{\infty} w(u+ct, x-u-ct) f(x) dx,$$

and

$$Q_{1,(1)}(u,t) = \lambda_2 e^{-(\lambda+\lambda_2+\delta)t} \int\limits_{u+ct}^{\infty} w(u+ct, x-u-ct) g(x) dx$$

$$+\lambda e^{-(\lambda+\lambda_2+\delta)t} \int\limits_{u+ct}^{\infty} w(u+ct, x-u-ct) f(x) dx.$$

- $N_1(t) \sim Poisson(\lambda = 1)$;
- $N_2(t) \sim Generalized\ Erlang(2)$ with $\lambda_1 = 0.5$ and $\lambda_2 = 1$;
- $X \sim Exp(1)$ and $Y \sim Exp(1)$;
- The premium rate $c = 1.5$;
- $w(x, y) = y$ for all x,y.

Values of $\mathbb{E}\left[\,|\,U(\tau)\,|\,\mathbb{I}(\tau < t)\,|\,U(0) = u\right]$

t	$u = 0$	$u = 5$	$u = 10$
0.25	0.197211	0.002292	0.000024
0.5	0.320497	0.005679	0.000085
0.75	0.402054	0.009953	0.000193
1	0.459419	0.014913	0.000357
1.25	0.502008	0.020387	0.000583
1.50	0.535017	0.026229	0.000876
1.75	0.560535	0.032326	0.001237
2	0.589942	0.038610	0.001669

Values of $\mathbb{E}\left[e^{-\delta\tau} \mid U(\tau) \mid \mathbb{I}(\tau < t) \mid U(0) = 10\right]$

t	$\delta = 0$	$\delta = 0.03$
0.25	0.00002493	0.00002482
0.5	0.00008558	0.00008476
0.75	0.00019331	0.00019046
1	0.00035706	0.00034993
1.25	0.00058328	0.00056860
1.50	0.00087614	0.00084959
1.75	0.00123792	0.00119412
2	0.00166929	0.00160184
2.25	0.00216907	0.00207048
2.50	0.00272503	0.00258562
2.75	0.00322125	0.00301355
3	0.00416771	0.00381874

- In practice, it is more likely that the surplus is checked at regular intervals and finite-time horizon ruin functions indicate that the company has to take action in order to make the business profitable.
- Realistic applications would require the computation of a finite-time Gerber-Shiu function.

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Thank you for your attention.